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LETTER TO THE EDITOR

Critical behaviour of driven bilayer systems: a field-theoretic renormalization group study

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Abstract
We investigate the static and dynamic critical behaviour of a uniformly driven bilayer Ising lattice gas at half-filling. Depending on the strength of the inter-layer coupling J, phase separation occurs across or within the two layers. The former transitions are controlled by the universality class of model A (corresponding to an Ising model with Glauber dynamics), with upper critical dimension $d_c = 4$. The latter transitions are dominated by the universality class of the standard (single-layer) driven Ising lattice gas, with $d_c = 5$ and a nonclassical anisotropy exponent. These two distinct critical lines meet at a nonequilibrium bicritical point which also falls into the driven Ising class. At all transitions, novel couplings and dangerous irrelevant operators determine corrections to scaling.

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1. Introduction

Driven diffusive lattice gases (DDLG), introduced by Katz *et al* [1] to investigate *far-from-equilibrium* properties of interacting many-particle systems, are deceptively simple generalizations of the familiar equilibrium Ising lattice gas [2]. Particles diffuse on a lattice, controlled by not only the usual inter-particle attraction and thermal bath (at temperature T), but also a *uniform external force*. In conjunction with periodic boundary conditions, the latter drives the system into a nonequilibrium steady state with nontrivial current. A remarkable range of novel collective phenomena emerge, some aspects of which are now well understood while many others remain mysterious [3]. In particular, when further 'slight' modifications or generalizations are introduced, simulations often contradict equilibrium-based expectations. In this sense, an intuitive understanding of nonequilibrium steady states is still lacking.

While most simulations of the original model were carried out on a single two-dimensional ($d = 2$) square lattice, Monte Carlo studies have been reported recently for two coupled driven

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Ising lattices stacked to form a bilayer structure [4–6]. Upon tuning the (Ising) *inter-layer* coupling, the system makes a transition from a homogeneous high-temperature phase into *two* distinct ordered states at low temperatures. A simple phase diagram was found [5], displaying two lines of continuous transitions which meet a line of 'first-order' transitions at a 'bicritical' point. The critical properties associated with the two lines were conjectured. In the most recent study [6], anisotropic *intra-layer* couplings were introduced, and certain critical properties were measured. In this letter, we report results from a field-theoretic renormalization group (RG) study for the continuous transitions of this model. Even though we essentially confirm the original conjecture [5], some novel and curious features emerge near the bicritical point and along the line associated with repulsive inter-layer couplings. After a brief description of both the microscopic model and the field-theoretic description, we present our results.

In all simulation studies, the 'microscopic' model consists of two fully periodic $L_1 \times L_2$ square lattices, arranged in a bilayer structure (effectively, an $L_1 \times L_2 \times 2$ system). The sites, labelled by (j_1, j_2, j_3), with $j_{1,2} = 1, \ldots, L_{1,2}$ and $j_3 = 1, 2$, may be empty or occupied. Thus, the set of occupation numbers $\{n(j_1, j_2, j_3)\}$, where $n = 0$ or 1, completely specifies a configuration. To access critical points, we use half-filled systems, i.e., $\sum n = L_1 L_2$. The particles interact, so the Hamiltonian is given by $\mathcal{H} \equiv -J_0 \sum nn' - J \sum nn''$, where n and n' are nearest neighbours *within* a given layer, while n and n'' differ only in the layer index. All studies focus on attractive intra-layer interactions, $J_0 > 0$, while J can be of either sign. Since intra-layer anisotropies generate no qualitatively new features, J_0 may be set to unity. The equilibrium phase diagram in the J–T plane is easily obtained, with some exactly known features. For example, a second-order transition occurs at $T_0 \simeq 0.5673/k_B$ and $J = 0$ [7].

To access the most interesting *nonequilibrium* properties, a *conserved dynamics* must be imposed. Typically, Kawasaki spin exchange with Metropolis rates is employed, i.e., particles hop to nearest-neighbour holes with probability $\min\{1, \exp(-\Delta\mathcal{H}/k_B T)\}$, where $\Delta\mathcal{H}$ is the energy change associated with the move. To model the effects of the drive, we add $\pm E_o$ to $\Delta\mathcal{H}$ for hops against/along, say, the 1-axis [1], interpreting the particles as 'charged' in the presence of an external 'electric' field $(E_o, 0, 0)$. Note that, with Metropolis rates, it is possible to study the 'infinite'-E_o case: jumps against the field are simply never executed. When so driven, the phase diagram in the T–J plane can be found (shown schematically in figure 1 [4–6]). At high T, the system is disordered (D). At low T, the system phase separates: for sufficiently repulsive J, the fully ordered ($T = 0$) state displays densities 1 and 0 in the two layers, so this phase is labelled 'full–empty' (FE). On the other hand, for $J > 0$ and low T, each layer phase separates individually, resulting in strips of particles (of width $L_2/2$, at $T = 0$) 'on top of each other', and aligned with the drive. Thus, we label this state the 'strip' (S) phase. In the following, we invoke field-theoretic RG techniques, to investigate *universal* properties associated with the critical points.

2. Model equations

This analysis has already been initiated in [3, 8]. We define the single-layer magnetizations $\varphi_i(\vec{x})$, $i = 1, 2$, as the coarse-grained versions of $2n(j_1, j_2, i) - 1$, with the in-layer coordinate $\vec{x}$ generalized to d dimensions. To ensure the proper $E_o = 0$ limit is obtained, we construct a Landau–Ginzburg–Wilson (LGW) Hamiltonian, $\mathcal{H}_c$, which contains all terms, up to fourth order in φ_i and second order in $\vec{\nabla}\varphi_i$, compatible with stability requirements and symmetries of the microscopic model. As usual, the explicit relationships [8] between the coarse-grained couplings and the microscopic parameters (J, J_0, T) are not needed. The next step is to incorporate the dynamics and the (coarse-grained) drive E ($\propto \tanh E_o$), following [3]. The result is a set of Langevin equations for the fields $\varphi_i(\vec{x}, t)$. Focusing on the deterministic

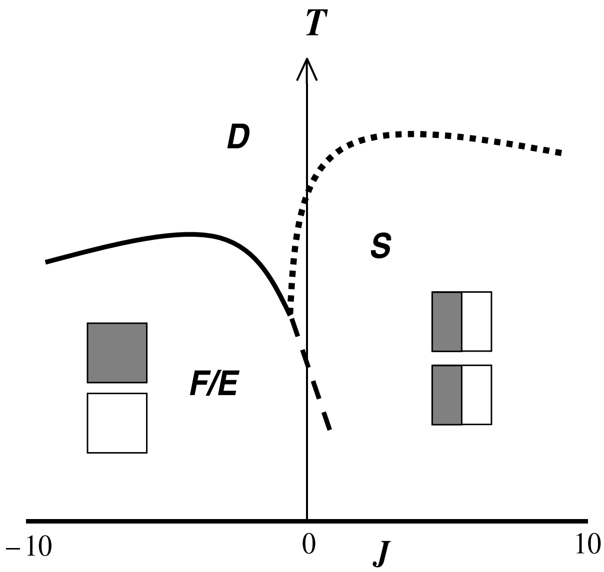

Figure 1. A schematic phase diagram in the J, T plane, for $E_o = \infty$. The D–FE (D–S) transition line is shown solid (dotted). The S and FE phases are separated by a (dashed) first-order line. The junction of the three lines marks the bicritical point.

evolution first, the basic equation for φ_1 (with a similar one for φ_2) reads

$$\partial_t \varphi_1 = \frac{\lambda}{2}(1 - \varphi_1\varphi_2)[\delta\mathcal{H}_c/\delta\varphi_2 - \delta\mathcal{H}_c/\delta\varphi_1] + \gamma[\nabla^2 (\delta\mathcal{H}_c/\delta\varphi_1) + \sqrt{2}E\,\partial\varphi_1^2].$$

Here, the first term models the energetics of *inter-plane* jumps, with a relaxation constant denoted by λ. The second term reflects the (conserved) *in-plane* dynamics, being the coarse-grained version of the usual (single-layer) DDLG, with diffusion coefficient γ. The symbol ∂ indicates a spatial gradient along the direction of the drive. Some numerical factors appear for later convenience. Anticipating the very different roles played by the total and 'staggered' magnetizations, we introduce $\Sigma(\vec{x}, t) \equiv [\varphi_1(\vec{x}, t) + \varphi_2(\vec{x}, t)]/\sqrt{2}$ and $\Delta \equiv [\varphi_1 - \varphi_2]/\sqrt{2}$. Finally, we incorporate the appropriate Langevin noise terms into the equations of motion, so the starting point of our field-theoretic analysis is

$$\partial_t \Sigma = \gamma\,\nabla^2\left[\left(r_\Sigma - \nabla^2\right)\Sigma + \frac{u}{6}\Sigma^3 + \frac{\tilde{u}}{2}\Sigma\Delta^2\right] + \gamma\,\partial\left(\frac{E}{2}\Sigma^2 + \frac{\bar{E}}{2}\Delta^2\right) + \zeta_\Sigma \tag{1}$$

$$\partial_t \Delta = -\lambda\left[\left(r_\Delta - \nabla^2\right)\Delta + \frac{g}{6}\Delta^3 + \frac{\tilde{g}}{2}\Delta\Sigma^2\right] + \lambda\left(\tilde{E}\Delta\,\partial\Sigma + \tilde{E}'\Sigma\,\partial\Delta\right) + \zeta_\Delta \tag{2}$$

with stochastic white noise characterized by $\langle\zeta_\Sigma\rangle = 0 = \langle\zeta_\Delta\rangle$, and correlations

$$\langle\zeta_\Sigma(\vec{x}, t)\zeta_\Sigma(\vec{x}', t')\rangle = -2\gamma\,\nabla^2\delta(\vec{x} - \vec{x}')\delta(t - t') \tag{3}$$

$$\langle\zeta_\Delta(\vec{x}, t)\zeta_\Delta(\vec{x}', t')\rangle = 2\lambda\delta(\vec{x} - \vec{x}')\delta(t - t'). \tag{4}$$

Some comments are in order. First, $\bar{E} = \tilde{E} = \tilde{E}' = E$ in mean field, but suffer different renormalizations, so our parameter space must be generalized. Second, for zero drive, the D–FE (D–S) transitions are marked by the vanishing of $r_\Delta(r_\Sigma)$, i.e., $\Delta(\Sigma)$ becoming critical. Third, the local Σ is conserved and unaffected by inter-layer jumps; hence, its relaxation is diffusive and independent of λ. Indeed, for $\tilde{u} = 0$ and vanishing drive, equation (1) reduces to that in model B dynamics [9]. Now, for nonzero drive, the terms containing only Σ fields are precisely the standard Langevin equation for the DDLG [10, 11]. Therefore, equation (1) is supplemented with a *conserved* stochastic noise term. On the other hand, for $\tilde{g} = 0$ and zero drive, equation (2) reduces to the purely dissipative dynamics of model A [9], controlled only by the inter-plane jump rate λ. Thus, *nonconserved* white noise has been added. The *cross* correlations $\langle \zeta_\Sigma \zeta_\Delta \rangle$ vanish, as a consequence of the statistical independence of intra- and inter-layer jumps. Finally, the drive will induce the standard DDLG spatial anisotropies, to be discussed next.

3. The D–S critical line

We begin with the D–S transitions, where the conserved field Σ becomes massless, while Δ remains noncritical. In the presence of the driving term, the characteristic anisotropies of DDLGs emerge. The system softens only in the spatial sector *transverse* to the drive, so transverse and longitudinal momenta scale differently: $q_\parallel \sim q_\perp^{1+\kappa}$, where κ is known as the anisotropy exponent ($\kappa = 1$ in mean field) [10, 11]. Consequently many longitudinal nonlinear terms become irrelevant. Next, since the nonconserved field Δ is massive, loop diagrams containing internal Δ propagators are *infrared*-convergent and may be ignored. Thus g and, remarkably, ζ_Δ may be neglected. Alternatively, the same conclusions can be reached by focusing on *ultraviolet* singularities in perturbation theory (see [12]). The result is that, within the Σ sector, the analysis is identical to that in the standard single-layer DDLG case [10, 11]. These considerations can be summarized in the effective theory:

$$\partial_t \Sigma = \gamma\big[c\,\partial^2 + \nabla_\perp^2\big(r_\perp - \nabla_\perp^2\big)\big]\Sigma + \gamma\frac{E}{2}\,\partial\Sigma^2 + \zeta_\Sigma \tag{5}$$

$$\partial_t \Delta = -\lambda\big(r_\Delta - \nabla^2\big)\Delta - \lambda\frac{\tilde{g}}{2}\Delta\Sigma^2 + \lambda\tilde{E}\Delta\,\partial\Sigma + \zeta_\Delta \tag{6}$$

where $c > 0$ is introduced to account for anomalous anisotropy [10, 11]. The D–S line itself is associated with a vanishing (renormalized) $r_\perp$, while $r_\Delta > 0$ (so $\nabla^2\Delta$ is also irrelevant). Denoting a momentum scale by μ ($\sim q_\perp$), a consistent set of naive dimensions emerges: $\gamma \sim \mu^0$, $\omega \sim \mu^4$, $\lambda \sim \mu^2$, and $c \sim \mu^0$. As in the standard case, the most relevant coupling is $\mathcal{E} = E^2/c^{3/2}$ ($\sim \mu^{5-d}$ naively), so the upper critical dimension is $d_c^\Sigma = 5$. In addition to c, only the coupling $\sigma = \tilde{g}c/E\tilde{E}$ exhibits a nontrivial flow under the RG. The remaining effective nonlinearities all have lower scaling dimension: u, $\tilde{u}$, $\tilde{E}\bar{E}$, $\tilde{E}'\bar{E} \sim \mu^{3-d}$, and $\tilde{E}\tilde{E}' \sim \mu^{1-d}$. Of course, the static coupling u drives the phase transition and therefore constitutes a dangerously irrelevant variable [10].

Remarkably, the analysis can be performed to all orders in $\varepsilon \equiv d_c^\Sigma - d$. In particular, the Σ loops which modify the new vertex ($\tilde{E}$) are the *same* as those affecting c. Thus, the same power series, $\rho(\mathcal{E})$, enters the RG flow equations for both renormalized couplings: $\beta_\mathcal{E} \equiv \mu\,\partial_\mu\mathcal{E} = -\mathcal{E}[\varepsilon + \frac{3}{2}\rho(\mathcal{E})]$ and $\beta_\sigma \equiv \mu\,\partial_\mu\sigma = \sigma(1+\sigma)\rho(\mathcal{E})$. For $d < 5$, the stable RG fixed point turns out to be $\rho(\mathcal{E}^*) = -2\varepsilon/3$ and $\sigma^* = -1$ to all orders, whereas for $d \geqslant 5$, both couplings tend to the Gaussian fixed point 0. Notice, however, that the novel bilayer coupling σ does not enter any singular diagram for the two-point functions. Consequently, despite its nontrivial fixed-point value, it does not affect the scaling exponents along the critical Σ line, which are just those of the standard DDLG. Even for $d < 5$, the only nontrivial exponent is κ. The two-point correlation function for the Σ fields scales as $C_{\Sigma\Sigma}(q, \omega, r_\perp) = q_\perp^{-6}\,\hat{C}_\Sigma(q_\perp r_\perp^{-1/2}, q_\parallel/q_\perp^{1+\kappa}, \omega/q_\perp^4)$, corresponding to the transverse critical exponents $\eta_\perp = 0$, $\nu_\perp = 1/2$, and $z_\perp = 4$. As a consequence of Galilean invariance [10], the exponent κ is fixed by a scaling relation to $1 + \varepsilon/3$ for $2 \leqslant d \leqslant 5$ ($\kappa = 1$ for $d \geqslant 5$).

4. The bicritical point

Remarkably, these features remain valid at the point where both fields are critical: $r_\Delta = r_\perp = 0$. Formally, this follows from the observation that the above scaling dimensions still hold, and no additional diagrams appear. The conserved field Σ essentially slaves the nonconserved field Δ. The larger critical dimension $d_c^\Sigma = 5$ still dominates the scaling behaviour, keeping the static couplings u, $\tilde{u}$, g, and $\tilde{g}$ irrelevant (with $\tilde{g}$ appearing merely through σ). Thus, there are no nontrivial renormalizations in the two-point function for the Δ fields, so its scaling is just *isotropic* and *mean-field*-like, $C_{\Delta\Delta}(q, \omega, r_\Delta) = q^{-4}\,\hat{C}_\Delta(q r_\Delta{}^{-1/2}, \omega/q^2)$. This corresponds to a nonconserved Gaussian theory: $\eta = 0$, $\nu = 1/2$, and $z = 2$ near five dimensions.

5. The D–FE critical line

Finally, we turn to the case where only the Δ field is critical: $r_\Delta \to 0$, $r_\perp > 0$. Though Σ remains massive, it is a conserved field and hence a slow variable. This situation is reminiscent of models E or G in equilibrium critical dynamics [9]. The key difference resides, however, in the external drive rendering the system far-from-equilibrium and manifestly anisotropic. At the same time, the coupling to the diffusive mode sets us outside the framework of the simple nonequilibrium kinetic Ising models investigated in [13]. So, a full RG calculation is required to determine whether model A relaxational kinetics still persists.

With Σ noncritical, we may safely neglect the terms $\nabla^4\Sigma$ and $\nabla^2\Sigma^3$. Naively, scaling is isotropic along this critical line, so $(q_\parallel, q_\perp) \sim \mu$, $\omega \sim \mu^2$, and $(\lambda, \gamma) \sim \mu^0$. The dominant nonlinear coupling is now $g \sim \mu^{4-d}$, with an upper critical dimension $d_c^\Delta = 4$. Power counting yields $\tilde{u} \sim \mu^{2-d}$ which therefore becomes irrelevant. However, beyond g, several additional marginal effective couplings appear, consisting of combinations such as $E\bar{E}$, $\tilde{E}\bar{E}$, $\tilde{E}'\bar{E}$, $\tilde{g}\bar{E}/E$. Hence the effective critical theory still contains *two* anisotropic propagators and *six* nonlinear vertices:

$$\partial_t \Sigma = \gamma\big(c\,\partial^2 + \nabla_\perp^2\big)\Sigma + \gamma\,\partial\left(\frac{E}{2}\Sigma^2 + \frac{\bar{E}}{2}\Delta^2\right) + \zeta_\Sigma \tag{7}$$

$$\partial_t \Delta = -\lambda\big(r_\Delta - a\,\partial^2 - \nabla_\perp^2\big)\Delta - \lambda\left(\frac{g}{6}\Delta^3 + \frac{\tilde{g}}{2}\Delta\Sigma^2\right) + \lambda\big(\tilde{E}\Delta\,\partial\Sigma + \tilde{E}'\Sigma\,\partial\Delta\big) + \zeta_\Delta. \tag{8}$$

Similar to the D–S line case, all diagrams that contain the conserved noise ζ_Σ are nonsingular. Thus, in contrast to, e.g., the case for model C, ζ_Σ becomes *irrelevant* here. Instead, the dynamics of the conserved field is dominated, via the coupling $\bar{E}$, by the fluctuating, nonconserved, critical Δ. This remarkable feature may be observable in simulations of the Σ–Σ correlations, which should develop anomalous (beyond the generic variety [14]) long-range components.

Carrying out an expansion in ϵ ($\equiv 4 - d$) at the one-loop level, we find 35 nontrivial Feynman graphs, complicated further by the two distinct anisotropies in Δ and Σ. For brevity, we only report the salient features and leave details to be published elsewhere [15]. The RG fixed points are determined from *seven* coupled RG flow equations for five marginal nonlinearities ($\bar{g} = g/a^{1/2}$, $f = E\bar{E}/a^{3/2}$, $\tilde{f} = \tilde{E}\bar{E}/a^{3/2}$, $\tilde{f}' = \tilde{E}'\bar{E}/a^{3/2}$, and $h = \tilde{g}\bar{E}/Ea^{1/2}$) and two dimensionless ratios ($v \equiv c/a$, $w \equiv \gamma/\lambda$). All equations involve (v, w) in algebraic expressions and, in general, fixed points (v^*, w^*) are not simply $O(\epsilon)$. Aside from the Gaussian fixed point, the most interesting nontrivial zeros of the RG β-functions are the *symmetric* fixed points ($\tilde{f}^* = \tilde{f}'^* = (fh/\bar{g}v)^* \neq 0$) and the fixed points restoring Galilean invariance ($h^* = 0$, $\tilde{f}'^* = w^* f^*$). Unfortunately, none are stable under the RG flow. The completely *stable* fixed points that we found are essentially in the class of model A: $\bar{g}^* = \epsilon/3 + O(\epsilon^2)$

and $\tilde{f}^* = \tilde{f}'^* = h^* = 0$. Remarkably, two unusual features emerge: first, (v^*, w^*) remain undetermined by the flow equations. Instead, they are constrained by the stability analysis alone, resulting in a stable fixed *domain* $\mathcal{D}$ [15]. Second, for each $(v^*, w^*) \in \mathcal{D}$, the coupling f^* assumes a nontrivial value

$$\epsilon \left[\frac{3w^{*2}}{(\sqrt{1 + v^* w^*} + \sqrt{1 + w^*})^2} - \frac{3w^*}{(1 + \sqrt{v^*})^2} \right]^{-1} + \mathrm{O}(\epsilon^2).$$

However, none of these novel aspects affect the leading exponents, as we summarize below.

Aside from technicalities, the features of the D–FE and D–S lines are similar: the critical exponents are determined by the critical field alone, while the noise of the massive mode becomes irrelevant. A single coupling survives, through which the critical field influences the dynamics of the massive mode, but not vice versa. Thus, on the D–FE line, we find standard model A behaviour: $C_{\Delta\Delta}(q, \omega, r) = q^{-z-2-\eta} \hat{C}_\Delta(q r_\Delta^{-\nu}, \omega/q^z)$, with $\eta = 0$, $\nu^{-1} = 2 - \epsilon/3$, and $z = 2$, up to $\mathrm{O}(\epsilon)$ (for $2 \leqslant d \leqslant 4$) and *trivial* anisotropic properties. Meanwhile, the leading singularities of the Σ field are of the simple diffusive type: $C_{\Sigma\Sigma}(q, \omega) = q_\perp^{-4} \hat{C}_\Sigma(\omega/q^2)$ (with no anomalous anisotropy: $\kappa = 0$). We stress, however, that unlike the usual kinetic Ising model, our bilayer system exhibits nonvanishing and anomalous *three-point* correlations [16]—an obvious signature of the external drive. These should be clearly detectable in simulations. In addition, corrections to scaling are expected to be distinct from those of model A.

6. Summary and discussion

To conclude, we find that the universal critical behaviour of a driven, bilayer Ising lattice gas is controlled by two distinct universality classes, depending on whether the system orders into S or FE configurations, i.e., whether the conserved sum, Σ, or the nonconserved difference, Δ, of the coarse-grained layer magnetizations becomes critical. In both cases, the fixed-point theory shows the critical field decoupling completely from the noncritical one, as the noise of the latter becomes irrelevant. Thus, all along the D–S line, the critical dimension is $d_c^\Sigma = 5$, and the scaling exponents for the critical Σ field are just those of the 'standard' DDLG. In contrast, for the D–FE transitions, the leading singularities are those of model A for the Δ field, with $d_c^\Delta = 4$ and isotropic exponents. At the bicritical point, though both Σ and Δ are critical, the larger critical dimension is $d_c^\Sigma = 5$, so the system again displays DDLG-like critical properties. However, at all critical points, a single nontrivial operator survives which provides a *one-way* coupling of the critical field into the dynamics of the noncritical one. For the D–S transitions, this operator is $\sigma = \tilde{g}c/E\tilde{E}$, while its counterpart at the D–FE transitions is $f = E\bar{E}/a^{3/2}$. Interestingly, both have *negative* fixed-point values, indicating that the three-point couplings do not all share the same sign. The stability domains of these fixed points are bounded, so trajectories starting outside the stable region can run off to infinity [15]. Even for stable trajectories, the presence of numerous other, unstable fixed points tends to enlarge crossover regimes, shrouding the asymptotic universal scaling behaviour. This may in fact lie at the core of the inconsistent exponent values observed in [6]. A better understanding of the bare couplings, associated with the microscopic Hamiltonian, would be extremely helpful.

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